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# Coulomb-gas formulation of $SU(2)$ branes and chiral blocks

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## Abstract

We construct boundary states in  $SU(2)_k$  WZNW models using the bosonized Wakimoto free-field representation and study their properties. We introduce a Fock space representation of Ishibashi states which are coherent states of bosons with zero-mode momenta (boundary Coulomb-gas charges) summed over certain lattices according to Fock space resolution of  $SU(2)_k$ . The Virasoro invariance of the coherent states leads to families of boundary states including the B-type D-branes found by Maldacena, Moore and Seiberg as well as the A-type corresponding to trivial current gluing conditions. We then use the Coulomb-gas technique to compute exact correlation functions of WZNW primary fields on the disk topology with A- and B-type Cardy states on the boundary. We check that the obtained chiral blocks for A-branes are solutions of the Knizhnik–Zamolodchikov equations.

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## 1. Introduction

Conformal field theory (CFT) on a group manifold provides an excellent test ground for the study of strings on curved backgrounds. Such models are exactly solvable due to large symmetries, while they are non-trivial enough to exhibit the peculiarity of curved target spaces. In recent years, D-branes in Wess–Zumino–Novikov–Witten (WZNW) models have been investigated by many authors, and remarkable progress has been made [1–12]. In particular, it is now understood that possible locations for stable branes are quantized, and the branes wrap conjugacy classes of the group manifold [13–15]. From the world-sheet point of view, branes are boundaries of a two-dimensional manifold. The world-sheet description is then a boundary CFT, where the branes correspond to specific states, called boundary states. The CFT analysis of D-branes is based on the algebraic formulation of boundary states for

rational conformal theories, dating back to the work of Cardy and Ishibashi [16, 17]. For rational models, operator product expansions involving boundary and various duality relations among the coupling constants were studied in [18–20]. Hence, at least for simple rational CFT models using the algebraic consistency conditions one can find which D-branes are allowed in a given closed-string background. It is an important open problem to generalize such analysis to more complicated theories, including non-rational models.

In this paper, we reconsider D-branes on WZNW backgrounds from a somewhat different perspective. CFT can also be formulated using the free-field (Coulomb-gas) representation [21–23], which is in a sense complementary to the algebraic approach. The free-field approach allows one to compute exact correlation functions in a constructive manner—sembling pieces and finding integral expressions—without having to solve complicated Knizhnik–Zamolodchikov equations [24]. Instead, one needs to work out the cohomology of the screen charges. The Coulomb-gas formalism has recently been extended to include boundary states, in the case of Virasoro minimal models [25, 26] and in the case of CFTs with  $W$ -algebra symmetries [27]. The present paper deals with a free-field formulation of boundary states on a WZNW background. This should provide a concrete representation of D-branes and should also be useful for computing exact correlation functions. Indeed, such a free-field description of branes for the  $U(1)_k$  WZNW model is well known [10, 28, 29]. The  $U(1)_k$  model corresponds to a free boson compactified on a circle of radius  $R = \sqrt{2k}$  (where  $k$  is a positive integer), allowing an extended symmetry so that the CFT becomes rational. The primary fields are then indexed by an integer  $r$  which is defined modulo  $2k$ . The boundary state construction starts from finding a basis, the Ishibashi states. They are coherent states of a bosonic field, with the ground state momenta summed over certain lattices. Corresponding to the  $2k$  primary fields of  $U(1)_k$ , there are [29]  $2k$  A-type Ishibashi states

$$|A; r\rangle_{U(1)} = \prod_{n>0} e^{\frac{1}{n}a_n\bar{a}_{-n}} \sum_{\ell \in \mathbb{Z}} \left| \frac{r+2k\ell}{\sqrt{2k}}, \frac{r+2k\ell}{\sqrt{2k}} \right\rangle, \quad (1)$$

where  $a_n$  and  $\bar{a}_n$  are holomorphic and anti-holomorphic Heisenberg operators; the ground states  $|\ast, \ast\rangle$  are parametrized by the left- and right-moving momenta (holomorphic and anti-holomorphic boundary charges). Our convention is  $r = 0, \dots, 2k - 1$ . By a T-duality transformation along the circle, the A-type states are related to  $2k$  B-type Ishibashi states

$$|B; r\rangle_{U(1)} = \prod_{n>0} e^{-\frac{1}{n}a_n\bar{a}_{-n}} \sum_{\ell \in \mathbb{Z}} \left| \frac{r+2k\ell}{\sqrt{2k}}, -\frac{r+2k\ell}{\sqrt{2k}} \right\rangle. \quad (2)$$

From the  $2k$  A-type Ishibashi states, one can construct  $2k$  A-type Cardy states

$$|\widetilde{A}; r\rangle_{U(1)} = (2k)^{-\frac{1}{4}} \sum_{s=0}^{2k-1} e^{-\pi i r s / k} |A; s\rangle_{U(1)}, \quad (3)$$

which are interpreted as D0-branes at  $2k$  special points on the circle. The B-type Cardy states are linear combinations of only two of the  $2k$  B-type Ishibashi states

$$|\widetilde{B}; \eta\rangle_{U(1)} = \left(\frac{k}{2}\right)^{\frac{1}{4}} (|B; 0\rangle_{U(1)} + \eta |B; k\rangle_{U(1)}), \quad (4)$$

where  $\eta = \pm 1$ . The other  $2k - 2$  B-type Ishibashi states decouple from the theory<sup>3</sup>. These B-type Cardy states are interpreted as D1-branes with a Wilson line parametrized by  $\eta$ . The

<sup>3</sup> In the standard language the A-type states defined above are Dirichlet and the B-type states are Neumann. It is, however, more natural to define them according to their coupling to bulk operators, and then the distinction of A- and B-branes depends on the convention of how we define the non-chiral (left  $\times$  right) bulk operators. This point is discussed in section 3.

main goal of this paper is to construct a similar free-field representation of the  $SU(2)_k$  branes by finding a Fock space representation of the boundary states for the parafermion part of  $SU(2)_k$ . In contrast to the  $U(1)_k$  part where the momenta are made periodic by the infinite sum over the identified values in (1) and (2), the momentum summation for the parafermion part is much more complicated. This is because the sum over momenta becomes a lattice sum with a non-trivial truncation of null vectors. Such null vector structures must be examined carefully in order that the boundary states satisfy the Cardy condition. We argue that for the  $SU(2)_k$  model the momenta should be summed over a lattice which is basically a Kaluza–Klein tower but modified according to the structure of null vectors; in the Coulomb-gas terminology such a lattice sum is over the genus one Felder complex arising from the Fock space resolution of the spectra on the torus.

The idea of realizing WZNW branes in the Wakimoto free-field representation is not new. The A-branes of the  $SU(2)_k$  model are constructed using the standard Wakimoto representation in [30] (see also [31]) and their BRST property is discussed in [32]. The novelty of this paper is a more general treatment, which naturally includes the B-branes of  $SU(2)_k$  found in [10] (see [29, 33–36] for general discussions), and the computational method to obtain exact correlation functions of bulk primary operators using the Coulomb-gas screening charges. For these purposes we shall employ a bosonic version of the Wakimoto construction, which renders the symmetry of the model more transparent.

The free-field realization of branes gives another technical tool for calculations—sometimes a more convenient one than the other approaches. For example, in the Coulomb-gas formalism correlation functions involving such branes may, in principle, be generalized to arbitrary topology by sewing smaller diagrams together; on the other hand solving Knizhnik–Zamolodchikov equations for higher topologies is notoriously difficult.

The plan of this paper is as follows. In the next section we review the bosonized Wakimoto free-field representation of WZNW models and collect necessary ingredients for the following discussions. In section 3 we construct boundary states of the  $SU(2)_k$  and the parafermion  $SU(2)_k/U(1)_k$  models using the free-field representation. We present sample computations of correlation functions involving the boundary states in section 4. We summarize the results and conclude in section 5.

## 2. Bosonic Wakimoto representation of the $SU(2)_k$ WZNW model

In this section we review the Coulomb-gas representation of the  $SU(2)_k$  WZNW model [37–47]. We follow the bosonic construction of [42–47] which emphasizes the fact that  $SU(2)_k$  can be written as a product of  $\mathbb{Z}_k$  parafermions and an  $U(1)_k$  boson. We shall spell out anti-holomorphic expressions explicitly when they differ from the holomorphic counterparts, as they will be necessary for discussing boundary states.

### 2.1. Wakimoto free-field representation and bosonization

We start from the Gauss decomposition of the group element  $g$  in  $SL(2, \mathbb{C})$ ,

$$g = \begin{pmatrix} 1 & \bar{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\bar{\varphi}} & 0 \\ 0 & e^{-\bar{\varphi}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}. \quad (5)$$

This leads to the Wakimoto representation of the  $SU(2)$  currents  $J = kg^{-1}\partial g$  and  $\bar{J} = -k(\bar{\partial}g)g^{-1}$ , that is [30, 47]

$$J^+ = \beta, \quad (6)$$

$$J^- = i\sqrt{2(k+2)}\partial\varphi\gamma - k\partial\gamma - \beta\gamma^2, \quad (7)$$

$$J^3 = -i\sqrt{\frac{k+2}{2}}\partial\varphi + \gamma\beta, \quad (8)$$

and

$$\bar{J}^+ = -i\sqrt{2(k+2)}\bar{\partial}\bar{\varphi}\bar{\gamma} + k\bar{\partial}\bar{\gamma} + \bar{\beta}\bar{\gamma}^2, \quad (9)$$

$$\bar{J}^- = -\bar{\beta}, \quad (10)$$

$$\bar{J}^3 = i\sqrt{\frac{k+2}{2}}\bar{\partial}\bar{\varphi} - \bar{\gamma}\bar{\beta}, \quad (11)$$

where the usual right-nested normal ordering is implicit. The fields  $\beta$  and  $\bar{\beta}$  have been introduced as auxiliary fields.  $\tilde{\varphi}$  has been rescaled and  $\varphi, \bar{\varphi}$  are respectively its holomorphic and anti-holomorphic part. These currents and Sugawara stress tensor generate the  $SU(2)$  affine Lie algebra

$$T(z)T(z') \sim \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial T(z')}{(z-z')}, \quad (12)$$

$$T(z)J^a(z') \sim \frac{J^a(z')}{(z-z')^2} + \frac{\partial J^a(z')}{(z-z')}, \quad (13)$$

$$J^a(z)J^b(z') \sim \frac{(k/2)\delta^{ab}}{(z-z')^2} + \frac{if^{abc}}{(z-z')}J^c(z'), \quad (14)$$

and likewise for the anti-holomorphic part, with central charge

$$c = \frac{3k}{k+2} \quad (15)$$

and structure constants  $f^{12}_3 = f^{23}_1 = f^{31}_2 = 1$ . The Wakimoto free-fields have operator products

$$\varphi(z)\varphi(z') \sim -\ln(z-z'), \quad \beta(z)\gamma(z') \sim \frac{-1}{z-z'}, \quad \gamma(z)\beta(z') \sim \frac{1}{z-z'}, \quad (16)$$

and similarly for  $\bar{\varphi}, \bar{\beta}$  and  $\bar{\gamma}$ . We bosonize the  $\beta$ - $\gamma$  system

$$\beta = -i\partial\chi e^{\eta-i\chi}, \quad (17)$$

$$\gamma = e^{-\eta+i\chi} \quad (18)$$

and redefine the fields by a linear transformation

$$\begin{aligned} \phi^{(1)} &= \varphi - i\sqrt{\frac{k+2}{2}}\eta - \sqrt{\frac{k+2}{2}}\chi, \\ \phi^{(2)} &= i\sqrt{\frac{k+2}{k}}\varphi + \frac{k+2}{\sqrt{2k}}\eta - i\sqrt{\frac{k}{2}}\chi, \\ \phi^{(3)} &= -\sqrt{\frac{k+2}{k}}\varphi + i\sqrt{\frac{2}{k}}\eta. \end{aligned} \quad (19)$$

Our convention for the operator products between the new bosonic fields is

$$\phi^{(i)}(z)\phi^{(j)}(z') \sim -\delta^{ij}\ln(z-z'), \quad (20)$$

where the indices  $i$  and  $j$  run 1, 2, 3. After the redefinition, the holomorphic parts of the chiral current take the form:

$$J^+(z) = -\sqrt{k}\Psi(z) e^{i\sqrt{\frac{2}{k}}\phi^{(3)}}, \quad J^-(z) = -\sqrt{k}\Psi^\dagger(z) e^{-i\sqrt{\frac{2}{k}}\phi^{(3)}}, \quad J^3(z) = i\sqrt{\frac{k}{2}}\partial\phi^{(3)}, \quad (21)$$

where  $\Psi(z)$  and  $\Psi^\dagger(z)$  are the parafermion currents,

$$\begin{aligned} \Psi(z) &= \frac{1}{\sqrt{2}} \left( -i\sqrt{\frac{k+2}{k}}\partial\phi^{(1)} + \partial\phi^{(2)} \right) e^{\sqrt{\frac{2}{k}}\phi^{(2)}}, \\ \Psi^\dagger(z) &= \frac{1}{\sqrt{2}} \left( -i\sqrt{\frac{k+2}{k}}\partial\phi^{(1)} - \partial\phi^{(2)} \right) e^{-\sqrt{\frac{2}{k}}\phi^{(2)}}. \end{aligned} \quad (22)$$

Thus the two bosons  $\phi^{(1)}$  and  $\phi^{(2)}$  constitute the bosonic representation of the  $\mathbb{Z}_k$  parafermions, whereas the  $U(1)_k$  part is represented by  $\phi^{(3)}$ .

The stress tensor takes the form:

$$T(z) = -\frac{1}{2}\delta_{ij}\partial\phi^{(i)}\partial\phi^{(j)} + 2i\alpha_0\partial^2\phi^{(1)} = -\frac{1}{2}\partial\phi \cdot \partial\phi + 2i\alpha_0\rho \cdot \partial^2\phi, \quad (23)$$

where

$$\alpha_0 = \frac{1}{2\sqrt{2(k+2)}} \quad (24)$$

and  $\rho = (1, 0, 0)$ .

Bosonization of the anti-holomorphic fields

$$\bar{\beta} = i\bar{\partial}\bar{\chi} e^{\bar{\eta}+i\bar{\chi}}, \quad (25)$$

$$\bar{\gamma} = e^{-\bar{\eta}-i\bar{\chi}}, \quad (26)$$

and redefinition of the fields

$$\begin{aligned} \bar{\phi}^{(1)} &= \bar{\varphi} - i\sqrt{\frac{k+2}{2}}\bar{\eta} + \sqrt{\frac{k+2}{2}}\bar{\chi}, \\ \bar{\phi}^{(2)} &= i\sqrt{\frac{k+2}{k}}\bar{\varphi} + \frac{k+2}{\sqrt{2k}}\bar{\eta} + i\sqrt{\frac{k}{2}}\bar{\chi}, \\ \bar{\phi}^{(3)} &= -\sqrt{\frac{k+2}{k}}\bar{\varphi} + i\sqrt{\frac{2}{k}}\bar{\eta}, \end{aligned} \quad (27)$$

lead to bosonic expressions of the anti-holomorphic currents and the stress tensor

$$\begin{aligned} \bar{J}^+(\bar{z}) &= \left( -i\sqrt{\frac{k+2}{2}}\bar{\partial}\bar{\phi}^{(1)} - \sqrt{\frac{k}{2}}\bar{\partial}\bar{\phi}^{(2)} \right) e^{-\sqrt{\frac{2}{k}}(\bar{\phi}^{(2)}+i\bar{\phi}^{(3)})}, \\ \bar{J}^-(\bar{z}) &= \left( -i\sqrt{\frac{k+2}{2}}\bar{\partial}\bar{\phi}^{(1)} + \sqrt{\frac{k}{2}}\bar{\partial}\bar{\phi}^{(2)} \right) e^{\sqrt{\frac{2}{k}}(\bar{\phi}^{(2)}+i\bar{\phi}^{(3)})}, \\ \bar{J}^3(\bar{z}) &= -i\sqrt{\frac{k}{2}}\bar{\partial}\bar{\phi}^{(3)}, \end{aligned} \quad (28)$$

$$\bar{T}(\bar{z}) = -\frac{1}{2}\bar{\partial}\bar{\phi} \cdot \bar{\partial}\bar{\phi} + 2i\alpha_0\rho \cdot \bar{\partial}^2\bar{\phi}. \quad (29)$$

From the form of the stress tensor (29), it is obvious that flipping and rotating the bosonic fields in the directions of  $\bar{\phi}^{(2)}$  and  $\bar{\phi}^{(3)}$  leave the stress tensor invariant. To be more specific,

we define transformations  $\omega$  of the anti-holomorphic bosonic fields  $\bar{\phi}^{(i)}$  which operate trivially on  $\bar{\phi}^{(1)}$  but linearly transform the remaining two components

$$\omega : \begin{pmatrix} \bar{\phi}^{(1)} \\ \bar{\phi}^{(2)} \\ \bar{\phi}^{(3)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \tilde{M} & \end{pmatrix} \begin{pmatrix} \bar{\phi}^{(1)} \\ \bar{\phi}^{(2)} \\ \bar{\phi}^{(3)} \end{pmatrix}, \quad (30)$$

where  $\tilde{M}$  is a  $2 \times 2$  matrix. The invariance of  $\bar{T}(\bar{z})$  implies that  $\tilde{M}$  must be orthogonal<sup>4</sup>,  $\tilde{M}^T \tilde{M} = 1$ . Under the transformations  $\omega$ , the anti-holomorphic  $SU(2)$  currents are not invariant. We shall denote the transformed currents as

$$\Omega \bar{J}^a(\bar{z}) = \bar{J}^a(\omega \bar{\phi}^{(i)}(\bar{z})). \quad (31)$$

That is,  $\Omega \bar{J}^a(\bar{z})$  are the currents constructed as (28) but with the transformed bosonic fields  $\omega \bar{\phi}^{(i)}$  instead of  $\bar{\phi}^{(i)}$ . Note that

$$\Omega \bar{J}^\pm = \bar{J}^\pm, \quad \Omega \bar{J}^3 = \bar{J}^3 \quad (32)$$

for  $\tilde{M} = \text{diag}(1, 1)$  and

$$\Omega \bar{J}^\pm = \bar{J}^\mp, \quad \Omega \bar{J}^3 = -\bar{J}^3 \quad (33)$$

for  $\tilde{M} = \text{diag}(-1, -1)$ . In general,  $\Omega \bar{J}^a$  cannot be written in a simple form.

The chiral  $SU(2)$  primary fields  $\Phi_{j,m}(z)$  with isospin  $j$  and magnetic quantum number  $m$  are represented by the vertex operators

$$V_{j,m}(z) = K_{j,m} \exp(i\alpha_{j,m} \cdot \phi(z)), \quad (34)$$

where  $K_{j,m} = [(2j)!/(j+m)!(j-m)!]^{1/2}$  and

$$\alpha_{j,m} = \left( -j\sqrt{\frac{2}{k+2}}, -im\sqrt{\frac{2}{k}}, m\sqrt{\frac{2}{k}} \right). \quad (35)$$

Using operator products with the stress tensor, their conformal dimensions are verified to be

$$h_{j,m} = \frac{1}{2} \alpha_{j,m} \cdot (\alpha_{j,m} - 4\alpha_0\rho) = \frac{j(j+1)}{k+2}. \quad (36)$$

The same primary field  $\Phi_{j,m}(z)$  can also be represented by another vertex operator

$$V_{j,m}^\dagger(z) = K_{j,m} \exp(i\alpha_{j,m}^\dagger \phi(z)), \quad (37)$$

where the conjugate charge  $\alpha_{j,m}^\dagger$  is

$$\alpha_{j,m}^\dagger = 4\alpha_0\rho - \alpha_{j,-m} = \left( (1+j)\sqrt{\frac{2}{k+2}}, -im\sqrt{\frac{2}{k}}, m\sqrt{\frac{2}{k}} \right). \quad (38)$$

The equivalence of  $V_{j,m}(z)$  and  $V_{j,m}^\dagger(z)$  is a basic feature of the Coulomb-gas formalism. It can be used to minimize the number of screening operators required for non-vanishing correlation functions. In the  $SU(2)_k$  model this is related to the equivalence of  $\Phi_{j,m}(z)$  and  $\Phi_{\frac{k}{2}-j, \frac{k}{2}+m}(z)$  which is sometimes called spectral flow identification.

Non-chiral (left  $\times$  right) vertex operators are simply direct products of chiral operators, but since there are two equivalent vertex operator representations for a single chiral field, there are four ways to express a non-chiral primary field  $\Phi_{j,m}(z, \bar{z})$ :

$$\begin{aligned} V_{j,m}(z) \bar{V}_{j,m}(\bar{z}), & \quad V_{j,m}(z) \bar{V}_{j,m}^\dagger(\bar{z}), \\ V_{j,m}^\dagger(z) \bar{V}_{j,m}(\bar{z}), & \quad V_{j,m}^\dagger(z) \bar{V}_{j,m}^\dagger(\bar{z}), \end{aligned} \quad (39)$$

where the right-moving part of the vertex operators are

$$\bar{V}_{j,m}(\bar{z}) = K_{j,m} \exp(i\alpha_{j,m} \cdot \bar{\phi}(\bar{z})), \quad \bar{V}_{j,m}^\dagger(\bar{z}) = K_{j,m} \exp(i\alpha_{j,m}^\dagger \cdot \bar{\phi}(\bar{z})). \quad (40)$$

<sup>4</sup> It may be more natural to consider  $\omega$  as *antilinear* transformations and  $\tilde{M}$  (and also  $M$  in section 3) being (anti-)unitary, so that the rotation between  $\bar{\phi}^{(2)}$  and  $\bar{\phi}^{(3)}$  is interpreted as  $SU(2)$  rotation within conjugacy classes. This should lead to a different choice of basis in the Hilbert space of the anti-holomorphic sector (see section 5).

2.2. Truncation of Fock modules and characters

The key element of the free-field formalism is the screening operators which control the structure of singular vectors. They are used for finding integral expressions of correlation functions, and are also used to construct Felder’s BRST operators whereby physical spectra of CFT are realized as their cohomology spaces [42, 48–51].

We focus on the parafermion part of  $SU(2)_k$ , as the  $U(1)_k$  part is trivial. The parafermion primary fields  $\Phi_{l,n}^{\text{PF}}(z)$  labelled by two integers  $l = 0, 1, \dots, k$  and  $n = -k, -k + 1, \dots, k - 1$  are represented by vertex operators

$$V_{l,n}^{\text{PF}}(z) = K_{l,n}^{\text{PF}} \exp \left\{ -\frac{il}{\sqrt{2(k+2)}}\phi^{(1)} + \frac{n}{\sqrt{2k}}\phi^{(2)} \right\}, \tag{41}$$

for  $-l \leq n \leq l$ . The normalization constant is  $K_{l,n}^{\text{PF}} = [l!/((l+n)/2)!((l-n)/2)!]^{1/2}$ . Viewing the parafermions independently, as opposed to as a part of  $SU(2)_k$ , there are also operators  $\Phi_{l,n}^{\text{PF}}(z)$  with  $n > l$ . They are obtained by successive application of  $\Psi(z)$  on (41). Acting on the vacuum, the vertex operators  $V_{l,n}^{\text{PF}}$  generate a state space, the Fock module  $F_{l,n}$ .

There are three screening operators

$$Q_1 = \oint dz V_1(z), \quad Q_{\pm} = \oint dz V_{\pm}(z), \tag{42}$$

where

$$V_1(z) = \partial\phi^{(2)} \exp \left\{ i\sqrt{\frac{2}{k+2}}\phi^{(1)} \right\}, \tag{43}$$

$$V_{\pm}(z) = \exp \left\{ -i\sqrt{\frac{k+2}{2}}\phi^{(1)} \pm \sqrt{\frac{k}{2}}\phi^{(2)} \right\}. \tag{44}$$

These screening operators satisfy the relations

$$Q_+^2 = Q_-^2 = 0, \quad Q_+Q_- + (-1)^k Q_-Q_+ = 0, \quad Q_{\pm}Q_1 + Q_1Q_{\pm} = 0. \tag{45}$$

The screening operators act on the Fock module  $F_{l,n}$  as

$$Q_{\pm}F_{l,n} = F_{l+k+2,n\pm k}, \quad Q_r F_{l,n} = F_{l-2r,n}, \tag{46}$$

where

$$Q_r = \frac{1}{r} \frac{e^{\frac{2\pi i r}{k+2}} - 1}{e^{\frac{2\pi i}{k+2}} - 1} \oint dz_1 \cdots \oint dz_r V_1(z_1) \cdots V_1(z_r), \tag{47}$$

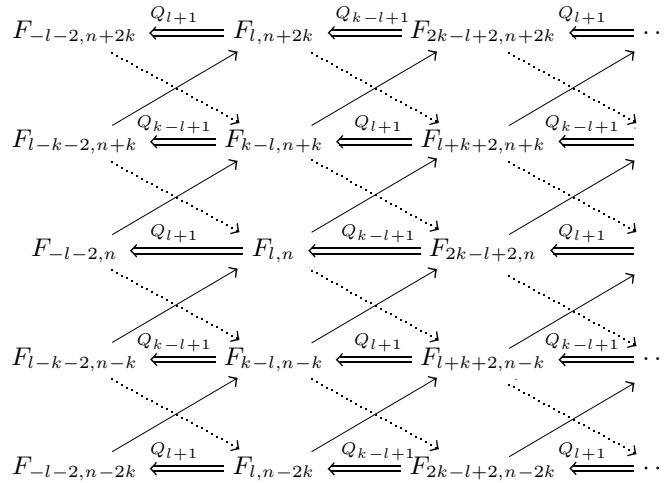
with the contours following Felder’s convention [48]. Repeated application of these operators on the Fock modules generates an infinite diagram  $\mathcal{C}_{\text{PF}}$  (figure 1). The physical spectrum in the parafermion module is expected to be realized by the cohomology space

$$\frac{\text{Ker}Q_+ \cap \text{Ker}Q_-}{\text{Im}Q_+Q_-} \cap \frac{\text{Ker}Q_{l+1}}{\text{Im}Q_{k-l+1} \cap \text{Ker}Q_{l+1}}. \tag{48}$$

The characters (string functions) of the parafermion modules are calculated in [42, 44, 46]. By assuming that the cohomology of  $(\text{Ker}Q_+ \cap \text{Ker}Q_-)/\text{Im}Q_+Q_-$  is trivial on the Fock module  $F_{r,s}$  except for  $(r, s) = (l, n)$ , the characters are found to be

$$\chi_{l,n}^{\text{PF}}(\tau) = \text{Tr}_{\frac{\text{Ker}Q_+ \cap \text{Ker}Q_-}{\text{Im}Q_+Q_-}(F_{l,n})} q^{L_0 - \frac{c}{24}} = \frac{1}{\eta(\tau)^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2 \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |n_2|, -n_1 > |n_2|}} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{(l+1+2(k+2)n_1)^2}{4(k+2)} - \frac{(n+2kn_2)^2}{4k}}, \tag{49}$$





**Figure 1.** A part of the infinite diagram  $C_{PF}$  generated by  $Q_{l+1}$ ,  $Q_{k-l+1}$  and  $Q_{\pm}$  for the parafermion module. The solid and dotted arrows indicate the operations with  $Q_+$  and  $Q_-$ , respectively.

where  $q = \exp(2\pi i\tau)$ . The lattice sum is made for both  $n_1 \geq |n_2|$  and  $n_1 < -|n_2|$  wedges, and we define  $\text{sign}(0) = +1$ . This expression agrees with the parafermion character formula obtained by other means [52, 53].

The  $SU(2)_k$  character for the isospin  $j$  representation is constructed from the parafermion characters as [10, 42, 54]

$$\chi_j^{SU(2)}(\tau) = \frac{1}{\eta(\tau)} \sum_n \sum_{n_3 \in \mathbb{Z}} \chi_{2j, n}^{PF} q^{k(n_3 + \frac{n}{2k})^2}. \tag{50}$$

The equivalence of this expression and the well-known  $SU(2)_k$  character formula

$$\chi_j^{SU(2)} = \frac{\Theta_{2j+1, k+2} - \Theta_{-2j-1, k+2}}{\Theta_{1, 2} - \Theta_{-1, 2}}, \tag{51}$$

is shown for example in appendix A of [42]. We use the Dedekind eta and Jacobi theta functions defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{52}$$

$$\Theta_{\lambda, \mu}(\tau) = \sum_{n \in \mathbb{Z}} q^{(2\mu n + \lambda)^2 / 4\mu}. \tag{53}$$

### 3. Fock space representation of boundary states

The boundary states for D-branes in WZNW models have to satisfy consistency conditions. One of them is the Ishibashi condition, i.e. the conservation of energy momentum across the boundary. Second, they must give appropriate overlaps for bulk chiral representations of the CFT. In this section we find the boundary states satisfying these conditions for the  $SU(2)_k$  model. A nice feature of working in the free-field Coulomb-gas approach is that we can start from a coherent state ansatz, as in the case of string theory in a flat background [55–57] (for an introductory review, see, e.g. [58, 59]).

### 3.1. Construction of $A$ -type $SU(2)_k$ boundary states

We consider the (non-chiral) Fock spaces of the bosons  $\phi^{(i)}$  and  $\bar{\phi}^{(i)}$  discussed in the previous section. The mode operators are defined through the expansion

$$\phi^{(i)}(z) = \varphi_0^{(i)} - i a_0^{(i)} \ln z + i \sum_{n \neq 0} \frac{a_n^{(i)}}{n} z^{-n}, \quad (54)$$

where  $z$  is a complex coordinate of the full plane. We assume the standard radial quantization on the  $z$ -plane and the mode operators satisfy the Heisenberg algebra

$$[a_m^{(i)}, a_n^{(j)}] = m \delta^{ij} \delta_{m+n,0}, \quad [\varphi_0^{(i)}, a_0^{(j)}] = i \delta^{ij}. \quad (55)$$

The anti-holomorphic counterpart  $\bar{\phi}^{(i)}(\bar{z})$  is expanded similarly, and the mode operators satisfy the same algebra. The holomorphic and anti-holomorphic Heisenberg operators are assumed to be independent. In particular,  $[\varphi_0^{(i)}, \bar{a}_0^{(j)}] = [\bar{\varphi}_0^{(i)}, a_0^{(j)}] = 0$ . We denote the ground states of the Fock spaces as  $|\alpha, \bar{\alpha}; \alpha_0\rangle = |\alpha^{(i)}, \bar{\alpha}^{(i)}; \alpha_0\rangle$ , which are annihilated by the positive modes  $a_{n>0}^{(i)}, \bar{a}_{n>0}^{(i)}$ , and  $\alpha^{(i)}$  and  $\bar{\alpha}^{(i)}$  are eigenvalues of the zero-mode momenta

$$a_0^{(i)} |\alpha, \bar{\alpha}; \alpha_0\rangle = \alpha^{(i)} |\alpha, \bar{\alpha}; \alpha_0\rangle, \quad \bar{a}_0^{(i)} |\alpha, \bar{\alpha}; \alpha_0\rangle = \bar{\alpha}^{(i)} |\alpha, \bar{\alpha}; \alpha_0\rangle. \quad (56)$$

These ground states are regarded as being constructed on the  $SL(2, \mathbb{C})$  invariant vacuum  $|0, 0; \alpha_0\rangle$  ( $\alpha_0$  is to remind the existence of the non-trivial background charge) by applying the vertex operators

$$|\alpha, \bar{\alpha}; \alpha_0\rangle = \lim_{z, \bar{z} \rightarrow 0} V_\alpha(z) \bar{V}_{\bar{\alpha}}(\bar{z}) |0, 0; \alpha_0\rangle = e^{i\alpha \cdot \varphi_0} e^{i\bar{\alpha} \cdot \bar{\varphi}_0} |0, 0; \alpha_0\rangle, \quad (57)$$

where

$$V_\alpha(z) = e^{i\alpha \cdot \phi(z)}, \quad \bar{V}_{\bar{\alpha}}(\bar{z}) = e^{i\bar{\alpha} \cdot \bar{\phi}(\bar{z})}. \quad (58)$$

The Fock spaces are generated on these ground states by operating with the negative mode operators  $a_{n<0}^{(i)}$  and  $\bar{a}_{n<0}^{(i)}$ . The corresponding bra ground states are given by

$$\langle \alpha, \bar{\alpha}; \alpha_0 | = \langle 0, 0; \alpha_0 | e^{-i\alpha \cdot \varphi_0} e^{-i\bar{\alpha} \cdot \bar{\varphi}_0}. \quad (59)$$

Note the difference of signs in the exponents of (57) and (59). The coefficients of  $i\varphi_0$  ( $i\bar{\varphi}_0$ ) in the exponents are the left-moving (right-moving) zero-mode momenta (holomorphic (anti-holomorphic) Coulomb-gas charges in the Coulomb-gas language) and are subject to momentum conservation (charge neutrality condition in Coulomb-gas). The ket states  $|\alpha, \bar{\alpha}; \alpha_0\rangle$  are thus interpreted to have momenta  $\alpha$  and  $\bar{\alpha}$ , whereas the momenta of the bra states  $\langle \alpha, \bar{\alpha}; \alpha_0 |$  are  $-\alpha$  and  $-\bar{\alpha}$ . With these definitions the momentum conservation demands the ground states being orthogonal, and we normalize them as  $\langle \alpha, \bar{\alpha}; \alpha_0 | \beta, \bar{\beta}; \alpha_0 \rangle = \delta_{\alpha\beta} \delta_{\bar{\alpha}\bar{\beta}}$ . The value of  $\alpha_0$  is common since it is fixed by (24).

Expanding the stress tensor (23) in powers of  $z$ , we find the Virasoro operators

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m \cdot a_{n-m} : - 2\alpha_0(n+1)\rho \cdot a_n. \quad (60)$$

We look for states<sup>5</sup>  $|B\rangle$  satisfying the Ishibashi condition

$$(L_n - \bar{L}_{-n})|B\rangle = 0, \quad (61)$$

in the Fock spaces introduced above. We make a coherent state ansatz [27]

$$|B(\alpha, \bar{\alpha}, \Lambda)\rangle = C_\Lambda |\alpha, \bar{\alpha}; \alpha_0\rangle, \quad (62)$$

<sup>5</sup> The  $B$  stands for *boundary* here. We hope no confusion arises with the  $B$ -type boundary states.

$$C_\Lambda = \prod_{m>0} \exp\left(\frac{1}{m} a_{-m} \cdot \Lambda \cdot \bar{a}_{-m}\right), \tag{63}$$

where  $\Lambda$  acts on the three components of the bosonic field as a  $3 \times 3$  matrix. Explicit calculation using the Baker-Campbell-Hausdorff formula shows that the boundary condition (61) implies

$$\Lambda^T \cdot \Lambda = I, \tag{64}$$

$$\Lambda \cdot \rho + \rho = 0, \tag{65}$$

$$\Lambda^T \cdot \alpha + 4\alpha_0 \rho - \bar{\alpha} = 0. \tag{66}$$

From (64) and (65), we find that  $\Lambda$  is of the form:

$$\Lambda = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & & M \\ 0 & & \end{pmatrix}, \tag{67}$$

where  $M$  is a  $2 \times 2$  orthogonal matrix. As the right-moving momenta are related to the left-moving ones through (66), we introduce an abbreviated notation of states

$$|B(\alpha, \Lambda)\rangle \equiv |B(\alpha, \Lambda^T \cdot \alpha + 4\alpha_0 \rho, \Lambda)\rangle, \tag{68}$$

where  $\Lambda$  is given by (67). The corresponding bra states are

$$\langle B(\alpha, \Lambda)| = \langle \alpha, \Lambda^T \cdot \alpha + 4\alpha_0 \rho; \alpha_0 | C_\Lambda^T, \tag{69}$$

where

$$C_\Lambda^T = \prod_{m>0} \exp\left(\frac{1}{m} a_m \cdot \Lambda \cdot \bar{a}_m\right). \tag{70}$$

The overlaps between these states are computed using the free-field representation as

$$\langle B(\alpha, \Lambda) | q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | B(\beta, \Lambda') \rangle = \frac{q^{\frac{1}{2}(\alpha - 2\alpha_0 \rho)^2 - \frac{1}{8}}}{\prod_{m>0} \det(1 - q^m \Lambda^T \cdot \Lambda')} \delta_{\alpha, \beta} \delta_{\Lambda^T \cdot \alpha, \Lambda'^T \cdot \beta}. \tag{71}$$

The states (68), (69) constructed above certainly satisfy the Ishibashi condition (61). However, they are not boundary states of the CFT. Having no restriction on  $\alpha$  means that some of the states are spurious. Physical states must be part of the cohomology of the screening operators, i.e. elements of the Felder complex. This is related to the requirement that boundary states must give overlaps which are partition functions for (irreducible) representations of the chiral algebra. Thus, we will instead consider the states  $|\Lambda; j\rangle$  for  $\Lambda$  of (67) and each isospin  $j$  of  $SU(2)$ , obtained by summing the charges  $\alpha$  over the Felder complex. Explicitly,

$$|\Lambda; j\rangle = \sum_{\alpha \in \Gamma_j} \kappa_{n_1} |B(\alpha, \Lambda)\rangle, \tag{72}$$

where  $\kappa_{n_1}$  is a phase factor related to its bra counterpart  $\kappa'_{n_1}$  through (75)<sup>6</sup> and the lattice  $\Gamma_j$  is

$$\begin{aligned} \Gamma_j : \alpha^{(1)} &= -j \sqrt{\frac{2}{k+2}} - n_1 \sqrt{2(k+2)}, \\ \alpha^{(2)} &= -im \sqrt{\frac{2}{k}} - in_2 \sqrt{2k}, \\ \alpha^{(3)} &= m \sqrt{\frac{2}{k}} + n_3 \sqrt{2k}, \end{aligned} \tag{73}$$

<sup>6</sup> These phase factors may be determined in principle by BRST invariance of the boundary states [32]. Our computation of correlation functions on disk topology below does not depend on such details since this factor can be absorbed in the vacuum normalization on the disk.

where

$$\begin{aligned} n_1, n_2 &\in \frac{1}{2}\mathbb{Z}, & n_3 &\in \mathbb{Z}, \\ n_1 - n_2 &\in \mathbb{Z}, \\ n_1 &\geq |n_2|, & -n_1 &> |n_2|, \\ m &\in \frac{1}{2}\mathbb{Z}, & 0 \leq m &\leq \frac{2k-1}{2}, & j - m &\in \mathbb{Z}. \end{aligned} \quad (74)$$

The lattice points of (73) contain a part indexed by  $(j, m)$ , equal to the labels (35) of the  $SU(2)$  primaries, and a part indexed by  $(n_1, n_2, n_3)$ , corresponding to periodic identifications. In fact, the  $\alpha^{(3)}$  sector is precisely the  $U(1)_k$  periodic lattice of (1). The lattice sums for  $n_1$  and  $n_2$  are those appeared in the parafermion character formula (49). The  $\alpha^{(1)}, \alpha^{(2)}$  sector is thus identified as the parafermion part.

The bra states  $\langle\langle \Lambda; j |$  are constructed similarly, but with a phase factor  $\kappa'_{n_1}$  in place of  $\kappa_{n_1}$ , satisfying

$$\kappa'_{n_1} \kappa_{n_1} = (-1)^{2n_1} \text{sign}(n_1), \quad (75)$$

where we define  $\text{sign}(0) = 1$  as before. With these definitions it is obvious from (71) and the character formulae (49), (50) that the overlaps between these states give the  $SU(2)_k$  characters

$$\langle\langle \Lambda; j | q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | \Lambda; j' \rangle\rangle = \chi_j^{SU(2)}(\tau) \delta_{jj'}. \quad (76)$$

The states  $| \Lambda; j \rangle$  are hence regarded as the Ishibashi states of the  $SU(2)_k$  model<sup>7</sup>.

These states are indexed by  $\Lambda$  as well as  $j$ .  $\Lambda$  is a generalization of the sign difference in (1) and (2) which distinguishes the A- and B-type  $U(1)_k$  Ishibashi states. So we expect that  $\Lambda$  specifies the type of the boundary states in the  $SU(2)_k$  theory. Indeed, with an explicit calculation using the bosonic expressions of the  $SU(2)$  currents (21) and (28), we can show that  $\Lambda$  is related to the current gluing conditions. In the present coordinates  $z, \bar{z}$ , the boundary is the unit circle  $z = \bar{z}^{-1}$ . Identifying

$$M = \tilde{M}, \quad (77)$$

we see that on the boundary the Ishibashi states satisfy

$$[zJ^\pm(z) - \bar{z}\Omega\bar{J}^\mp(\bar{z})] | \Lambda; j \rangle = 0, \quad (78)$$

$$[zJ^3(z) - \bar{z}\Omega\bar{J}^3(\bar{z})] | \Lambda; j \rangle = 0. \quad (79)$$

If we map the unit disc onto the upper half plane, the above conditions reduce to the standard form of the gluing automorphisms in the open string picture

$$J^\pm + \Omega\bar{J}^\mp = 0, \quad J^3 + \Omega\bar{J}^3 = 0. \quad (80)$$

Note that these gluing conditions reduce to

$$J^\pm + \bar{J}^\mp = 0, \quad J^3 + \bar{J}^3 = 0 \quad (81)$$

when  $M = \text{diag}(1, 1)$  and

$$J^\pm + \bar{J}^\pm = 0, \quad J^3 - \bar{J}^3 = 0 \quad (82)$$

when  $M = \text{diag}(-1, -1)$ , but cannot be written in simple forms in other cases.

In order to look into this in more detail, let us first focus on the A-type states of the  $SU(2)_k$  model, which are characterized by the trivial gluing conditions of the currents  $J^a$ .

<sup>7</sup> There remains some ambiguity due to the reflection symmetry of the Weyl group, which can be removed as [26, 30] or [60].

Consider the half-plane geometry where the A-type boundary condition is imposed on the real axis and use the mirroring of [61]. The trivial current gluing conditions imply that the anti-holomorphic currents  $\bar{J}^a$  (on the ‘lower half plane’) are analytical continuations of the holomorphic currents  $J^a$  on the upper half plane. This allows us to map the CFT on the half plane to a chiral CFT on the full plane. A  $p$ -point correlation function on the half plane is then equivalent to a  $2p$ -point function on the full plane. In particular, consider a one-point function of a primary field  $\Phi_{j,m}(z, \bar{z})$  on the upper half plane, which is written as a two-point function on the full plane:

$$\langle \Phi_{j,m}(w, \bar{w}) \rangle_{\text{UHP,A}} = \langle \Phi_{j,m}(w) \Phi_{j,-m}(w^*) \rangle_{\text{FP}}. \quad (83)$$

Let us evaluate the left-hand side in the free-field representation. We first map the upper half plane to the unit disc. Then we represent the primary operator  $\Phi_{j,m}(z, \bar{z})$  by  $V_{j,m}(z) \bar{V}_{j,m}^\dagger(\bar{z})$  using the free-field representations (39), and represent the boundary with the boundary state  $\langle\langle \Lambda; j |$ . The lhs is thus proportional to

$$\langle\langle \Lambda; j | V_{j,m}(z) \bar{V}_{j,m}^\dagger(\bar{z}) | 0, 0; \alpha_0 \rangle. \quad (84)$$

Note that we could, as well, have used one of the other vertex operator representations of (39). The other representations would also lead to the same result as below, but require insertion of screening operators, making the calculation a bit longer. The disc one-point function (84) vanishes unless the left- and right-moving momenta are separately conserved (modulo screening charges),

$$-\alpha + \alpha_{j,m} = 0, \quad (85)$$

$$-\Lambda^T \cdot \alpha - 4\alpha_0 \rho + \alpha_{j,m}^\dagger = 0, \quad (86)$$

where  $\alpha$  is one of the boundary momenta (Coulomb-gas charges) summed over in (72). It is obvious that the above conditions are satisfied for  $\alpha = \alpha_{j,m}$  only when  $\Lambda = \text{diag}(-1, 1, 1)$ . As the right-hand side of (83) is clearly non-vanishing, we conclude that A-type boundary states correspond to  $\Lambda = \text{diag}(-1, 1, 1)$ , and hence the A-type Ishibashi states in the  $SU(2)_k$  theory are written as

$$|A; j\rangle\rangle = |\Lambda = \text{diag}(-1, 1, 1); j\rangle\rangle \quad (87)$$

(in fact, there is some freedom left, which we explain in the next subsection). With these Ishibashi states the A-type Cardy states are constructed in the usual way as

$$|\widetilde{A}; j\rangle = \sum_{j'} \frac{S_{jj'}}{\sqrt{S_{0j'}}} |A; j'\rangle\rangle, \quad (88)$$

where  $S_{\lambda,\mu}$ ,  $\lambda, \mu = 0, \frac{1}{2}, \dots, \frac{k}{2}$ , is the  $SU(2)_k$  modular S-matrix,

$$S_{\lambda,\mu} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(2\lambda+1)(2\mu+1)}{k+2}\right). \quad (89)$$

### 3.2. One-point functions and rotation of branes

In the previous subsection we identified the A-branes by the condition (83), namely the universal coupling between the Ishibashi states and the bulk primary operators. The obtained states agree with what is known in the literature. In particular, one may reproduce overlaps between these states by explicit free-field calculations. The above representation for the A-branes (87) is, however, not general enough since the branes may be rotated within the group without changing their properties. We argue that such rotations are implemented in

the free-field representation by rotations between  $\bar{\phi}^{(2)}$  and  $\bar{\phi}^{(3)}$ , namely the transformation  $\omega$  of (30) which keeps the value of the  $\det \tilde{M}$  constant.

In deriving (87) we have used the convention that the non-chiral primary operators have equal (up to the conjugation  $\alpha_{j,m} \leftrightarrow \alpha_{j,m}^\dagger$ ) left- and right-momenta. This condition can be relaxed because instead of  $\bar{V}_\alpha(\bar{z})$  we may use rotated anti-holomorphic vertex operators

$$\Omega \bar{V}_\alpha(\bar{z}) \equiv \exp(i\alpha \cdot \omega \bar{\phi}(\bar{z})). \tag{90}$$

Now, upon identifying  $\tilde{M}$  of  $\omega$  and  $M$  of  $\Lambda$ , we can see that charge neutrality is satisfied in the one-point functions

$$\langle\langle \Lambda; j | V_{j,m}(z) \Omega \bar{V}_{j,m}^\dagger(\bar{z}) | 0, 0; \alpha_0 \rangle\rangle \tag{91}$$

for all  $\Lambda = -(1, M)$ . This indicates that bulk primary fields transformed by  $(1 \otimes \omega)$  always feel the states  $\langle\langle \Lambda; j |$  as A-branes. We may thus allow a rotation of the right-moving part of the bulk primary field and regard the branes rotated from (87), that is,  $|\Lambda; j\rangle\rangle$  with  $\det M = +1$ , as A-branes. In particular,  $\Lambda = \text{diag}(-1, -1, -1)$  also represents A-branes (with appropriately rotated bulk operators). We will use this in the next subsection.

### 3.3. Parafermion and B-type $SU(2)_k$ boundary states

In the  $SU(2)_k$  WZNW model there are known to be other kinds of branes called B-type, which are constructed from A-branes by orbifolding and T-duality [10]. Before discussing the B-branes in  $SU(2)_k$ , let us consider the A- and B-type boundary states of the parafermion model  $SU(2)_k/U(1)_k$ .

In the general  $SU(2)_k$  Ishibashi state formula (72), one could clearly identify a sector corresponding to the parafermions, and a sector corresponding to the  $U(1)_k$ . Stripping off the latter contribution then gives the Fock space representation of the A-type parafermion Ishibashi states

$$\begin{aligned} |A; j, n\rangle\rangle_{\text{PF}} &= \prod_{m>0} \exp \left\{ -\frac{1}{m} a_{-m}^{(1)} \bar{a}_{-m}^{(1)} + \frac{1}{m} a_{-m}^{(2)} \bar{a}_{-m}^{(2)} \right\} \\ &\times \sum_{\alpha^{(1)}, \alpha^{(2)} \in \Gamma_{j,n}^{\text{PF}}} \kappa_{n_1} |\alpha^{(1)}, \alpha^{(2)}, \bar{\alpha}^{(1)} = 4\alpha_{0\rho} - \alpha^{(1)}, \bar{\alpha}^{(2)} = \alpha^{(2)}; \alpha_0\rangle, \end{aligned} \tag{92}$$

where  $\kappa_{n_1}$  is the same phase factor as in the  $SU(2)_k$  case. The parafermion charge summation goes over the sublattice of (73),

$$\begin{aligned} \Gamma_{j,n}^{\text{PF}} : \alpha^{(1)} &= -j \sqrt{\frac{2}{k+2}} - n_1 \sqrt{2(k+2)}, \\ \alpha^{(2)} &= -in \frac{1}{\sqrt{2k}} - in_2 \sqrt{2k} \end{aligned} \tag{93}$$

where

$$\begin{aligned} n_1, n_2 &\in \frac{1}{2}\mathbb{Z}, \\ n_1 - n_2 &\in \mathbb{Z}, \\ n_1 &\geq |n_2|, \quad -n_1 > |n_2|. \end{aligned} \tag{94}$$

Consider then the expression in [10] for the A-type  $SU(2)_k$  Ishibashi states as a combination of the A-type parafermionic and  $U(1)_k$  states,

$$|A; j\rangle\rangle = \sum_{n=0}^{2k-1} \frac{1 + (-1)^{2j+n}}{2} |A; j, n\rangle\rangle_{\text{PF}} |A; n\rangle\rangle_{U(1)}. \tag{95}$$

If we substitute into this the expressions (92) and (1), we recover the formula (72). The projection  $\frac{1+(-1)^{2j+n}}{2}$  enforces the constraint  $j - m \in \mathbb{Z}$  of (74).

The B-type Ishibashi states in the parafermion theory are defined in [10] using an operator  $e^{i\pi\bar{J}_0^1}$  which changes the sign of the  $\bar{J}_0^3$  eigenvalue, as

$$(1 \otimes e^{i\pi\bar{J}_0^1})|A; j\rangle = \sum_{n=0}^{2k-1} \frac{1 + (-1)^{2j+n}}{2} |B; j, n\rangle_{\text{PF}} |B; n\rangle_{U(1)}. \quad (96)$$

In the free-field language, the action of the operator  $(1 + e^{i\pi\bar{J}_0^1})$  (which flips the sign of  $n$  in the above) is equivalent to replacing  $M$  by  $-M$ . Hence using the free-field representation (87), the left-hand side of (96) is the  $SU(2)_k$  A-type Ishibashi states with  $\Lambda = \text{diag}(-1, -1, -1)$ . Since the  $U(1)_k$  part of these states is clearly the B-type, we can mod it out according to (96) and find

$$\begin{aligned} |B; j, n\rangle_{\text{PF}} &= \prod_{m>0} \exp \left\{ -\frac{1}{m} a_{-m}^{(1)} \bar{a}_{-m}^{(1)} - \frac{1}{m} a_{-m}^{(2)} \bar{a}_{-m}^{(2)} \right\} \\ &\times \sum_{\alpha^{(1)}, \alpha^{(2)} \in \Gamma_{j,n}^{\text{PF}}} \kappa_{n_1} |\alpha^{(1)}, \alpha^{(2)}, \bar{\alpha}^{(1)} = 4\alpha_{0\rho} - \alpha^{(1)}, \bar{\alpha}^{(2)} = -\alpha^{(2)}, \alpha_0\rangle, \end{aligned} \quad (97)$$

where the lattice summation is the same as for the A-type states (93), (94). The Cardy states of the parafermion theory are constructed in the standard manner using the above free-field representation for the Ishibashi states.

Finally, the B-branes of the  $SU(2)_k$  theory are constructed from the A-type parafermion states and B-type  $U(1)_k$  states, or from the B-type parafermion states and A-type  $U(1)_k$  states [10]. For example, one may construct the B-type Cardy states of the  $SU(2)_k$  model as

$$\begin{aligned} |\widetilde{B}; j, \eta\rangle &= \sqrt{k} \sum_{j'=0}^{k/2} \frac{S_{jj'}}{\sqrt{S_{0j'}}} \left[ \frac{1 + (-1)^{2j'}}{2} |A; j', 0\rangle_{\text{PF}} |B; 0\rangle_{U(1)} \right. \\ &\quad \left. + (-1)^{2j} \eta \frac{1 + (-1)^{2j'+k}}{2} |A; j', k\rangle_{\text{PF}} |B; k\rangle_{U(1)} \right], \end{aligned} \quad (98)$$

where  $\eta = \pm 1$ . In the above case the states have  $\Lambda = \text{diag}(-1, 1, -1)$ , but in a similar construction from B-type parafermion states and A-type  $U(1)_k$  states we have  $\Lambda = \text{diag}(-1, -1, 1)$ . These two cases are connected by a rotation within the group, and hence B-type  $SU(2)_k$  boundary states are characterized by  $\det M = -1$ . In summary, boundary states with  $\det M = +1$  are considered as A-branes and  $\det M = -1$  as B-branes in the  $SU(2)_k$  model.

We comment on a subtlety arising in the distinction of A- and B-type branes. The A- and B-type boundary states constructed above have desired properties that A-type branes couple to bulk operators (with any magnetic quantum number  $m$ ) universally, whereas B-type branes couple only to operators with  $m = 0$  up to reflection symmetry. These are fundamental properties which distinguish A- and B-branes. However, we might as well use a (slightly unnatural) convention such that the bulk primary fields are constructed from a holomorphic part and its 'twisted' anti-holomorphic counterpart, that is, (90) with  $\tilde{M}$  connected to  $\text{diag}(+1, -1)$  by rotation. In this case the roles of branes with  $\det M = \pm 1$  are completely exchanged, and we must regard  $\det M = +1$  states as type B and  $\det M = -1$  as type A since the A- and B-branes are entirely symmetric (e.g. the overlaps between them) except for the coupling with bulk operators. In this sense the definitions of A- and B-branes are relative, and depend on how to define bulk non-chiral operators.

#### 4. Correlation functions, chiral blocks and boundary states

The Coulomb-gas computation of WZNW correlation functions was developed in [37, 38, 43, 47, 45] and was extended to higher topologies in [40, 42, 44, 46, 49]. In this section we apply this technique to the computation on the disc topology, where the boundary condition is represented by the boundary states which have been discussed in the previous section. As physical boundary conditions are realized by the Cardy states, a  $p$ -point function on the unit disc which we shall consider is given by

$$\langle \text{Cardy state} | V_1(z_1) \bar{V}_1(\bar{z}_1) \cdots V_p(z_p) \bar{V}_p(\bar{z}_p) (\text{screening operators}) | 0, 0; \alpha_0 \rangle, \quad (99)$$

where  $|0, 0; \alpha_0\rangle$  is the  $SL(2, \mathbb{C})$  invariant vacuum at the centre of the disc. The free-field representations of the boundary states, the primary operators and the screening operators have already been given. Thus the above expression can be straightforwardly evaluated in the free-field formalism, i.e. with repeated use of the Heisenberg algebra and the Baker–Campbell–Hausdorff formula.

A few technical remarks for the actual computation are in order [26, 27]. Firstly, we are allowed to insert different numbers of screening operators in holomorphic and anti-holomorphic sectors although this might seem odd from the mirroring (or doubling) picture of [61]. The reason for this is that the mirroring argument is based on the analytic continuation of the holomorphic and anti-holomorphic currents, but the screening operators have by construction trivial effects on the currents of the chiral algebra; in other words, the mirror does not see the screening operators. The second point is that, as the boundary has charges, the neutrality condition of the Coulomb-gas charges must now take the contributions from the boundary into account. The Cardy states constructed in the last section are linear sums of the states  $\langle B(\alpha, \Lambda) |$  which have a holomorphic charge  $-\alpha$  and an anti-holomorphic charge  $-(\Lambda^T \cdot \alpha + 4\alpha_0\rho)$  (recall that  $\langle \alpha, \bar{\alpha}; \alpha_0 |$  and  $|\alpha, \bar{\alpha}; \alpha_0\rangle$  have opposite charges). The correlation function (99) thus reduces to a sum of the amplitudes

$$\langle B(\alpha, \Lambda) | V_1(z_1) \bar{V}_1(\bar{z}_1) \cdots V_p(z_p) \bar{V}_p(\bar{z}_p) (\text{screening operators}) | 0, 0; \alpha_0 \rangle, \quad (100)$$

whose holomorphic and anti-holomorphic charges must be independently neutral:

$$-\alpha + \alpha_1 + \cdots + \alpha_p + (\text{holomorphic screening charges}) = 0, \quad (101)$$

$$-(\Lambda^T \cdot \alpha + 4\alpha_0\rho) + \bar{\alpha}_1 + \cdots + \bar{\alpha}_p + (\text{anti-holomorphic screening charges}) = 0. \quad (102)$$

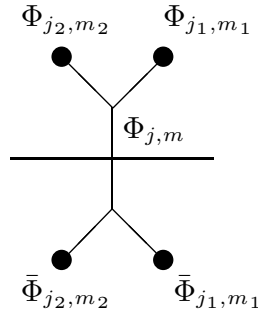
Otherwise the amplitudes vanish. The neutrality of charges corresponds to consistency of fusion rules among the primary operators [48], and non-vanishing amplitudes (100) are interpreted as chiral blocks [26].

Below we shall give sample calculations based on these observations. We focus on simple two-point functions of the  $SU(2)_k$  model and explain the method for both A- and B-type Cardy states.

##### 4.1. A-branes in $SU(2)_k$ model

Let us first consider a general two-point function of bulk primary operators  $\Phi_{j_1, m_1}(z_1, \bar{z}_1)$  and  $\Phi_{j_2, m_2}(z_2, \bar{z}_2)$  on the unit disc. On the boundary of the disc we assume the A-type Cardy states  $\langle A; j |$ . The calculation for the A-type boundary is entirely straightforward and parallel to the corresponding four-point case on the full plane (e.g. [47]). This, of course, is expected since the trivial current gluing conditions implied by the A-type boundary allow analytic continuation of the chiral currents to the full plane, and then the correlation function we are considering must satisfy the same Knizhnik–Zamolodchikov equations as for the chiral





**Figure 2.** Chiral block for a two-point function on the half plane. The intermediate state  $\Phi_{j,m}$  occurring as a fusion product of  $\Phi_{j_1,m_1}$  and  $\Phi_{j_2,m_2}$  couples to the corresponding Ishibashi state on the boundary.

four-point function on the full plane. We shall choose the vertex operator representation of the bulk operators as

$$\Phi_{j_1,m_1}(z_1, \bar{z}_1) : V_{j_1,m_1}(z_1) \bar{V}_{j_1,m_1}(\bar{z}_1), \tag{103}$$

$$\Phi_{j_2,m_2}(z_2, \bar{z}_2) : V_{j_2,m_2}(z_2) \bar{V}_{j_2,m_2}^\dagger(\bar{z}_2);$$

see section 2 for their definitions. We might choose other representations from the ones listed in (39) but they are equivalent (and involve unnecessarily many screening operators in following computations). Given holomorphic and anti-holomorphic boundary charges  $-\alpha$  and  $-\bar{\alpha} = -4\alpha_0\rho - \Lambda^T \cdot \alpha$  (note that we now have  $\Lambda = \text{diag}(-1, +1, +1)$  for the A-branes), the sums of the bulk and boundary Coulomb-gas charges in (100) are

$$\text{Holomorphic} : -\alpha + \alpha_{j_1,m_1} + \alpha_{j_2,m_2}, \tag{104}$$

$$\text{Anti-holomorphic} : -4\alpha_0\rho - \Lambda^T \cdot \alpha + \alpha_{j_1,m_1} + \alpha_{j_2,m_2}^\dagger. \tag{105}$$

Now we ask whether these charges are screenable, that is, whether it is possible to neutralize them by inserting screening operators. It is a simple exercise to check that the both sectors are neutralized by  $n$  holomorphic screening operators

$$Q_1 = \oint dz \partial\phi^{(2)} e^{i\sqrt{\frac{2}{k+2}}\phi^{(1)}}, \tag{106}$$

and  $(l - n)$  anti-holomorphic screening operators

$$\bar{Q}_1 = \oint d\bar{z} \bar{\partial}\bar{\phi}^{(2)} e^{i\sqrt{\frac{2}{k+2}}\bar{\phi}^{(1)}}, \tag{107}$$

where  $l = 2j_1$  and  $n = 0, 1, \dots, l$ . For each  $n$ , the charge  $\alpha$  on the boundary is found to be

$$\alpha = \left( (n - j_1 - j_2)\sqrt{\frac{2}{k+2}}, -i(m_1 + m_2)\sqrt{\frac{2}{k}}, (m_1 + m_2)\sqrt{\frac{2}{k}} \right). \tag{108}$$

Note that the values of  $\alpha$  reflect the fusion of the  $SU(2)_k$  primary fields

$$\Phi_{j_1,m_1} \times \Phi_{j_2,m_2} = \sum_{\substack{j=|j_1-j_2| \\ j_1+j_2+j \in \mathbb{Z}}}^{\min(j_1+j_2, k-j_1-j_2)} \Phi_{j,m=m_1+m_2}. \tag{109}$$

The neutrality of charges implies that the resulting intermediate state  $\Phi_{j,m}$  couples to the boundary, only via the corresponding Ishibashi state  $\langle\langle A; j \rangle\rangle$  (figure 2). Using the expression

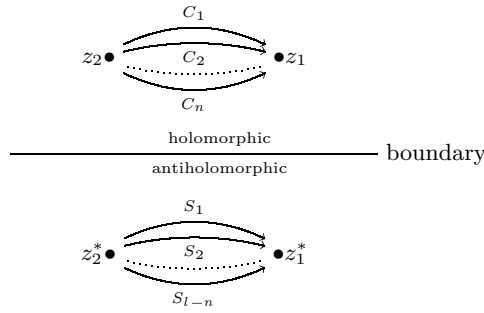


Figure 3. Integration contours of chiral blocks for A-branes.

of the A-type Cardy states (88), the two-point function is written as a linear combination of chiral blocks

$$\langle \Phi_{j_1, m_1}(z_1, \bar{z}_2) \Phi_{j_2, m_2}(z_2, \bar{z}_2) \rangle_{\text{disc}, \tilde{A}, j} = \frac{\sqrt{S_{00}}}{S_{j_0}} \sum_{j'} \frac{S_{jj'}}{\sqrt{S_{0j'}}} I_{j', m}^A, \tag{110}$$

where  $m = m_1 + m_2$  and

$$I_{j', m}^A = \langle B(\alpha, \Lambda) | V_{j_1, m_1}(z_1) \bar{V}_{j_1, m_1}(\bar{z}_1) V_{j_2, m_2}(z_2) \bar{V}_{j_2, m_2}^\dagger(\bar{z}_2) Q_1^n \bar{Q}_1^{l-n} | 0, 0; \alpha_0 \rangle. \tag{111}$$

The overall factor  $\sqrt{S_{00}}/S_{j_0} = \langle \tilde{A}; j | 0, 0; \alpha_0 \rangle^{-1}$  comes from the normalization of the vacuum. The boundary charge  $\alpha$  is as in (108) and  $n = j_1 + j_2 - j'$ ,  $\Lambda = \text{diag}(-1, +1, +1)$ . The screening charges yield  $n$ -tuple integration in the holomorphic and  $(l - n)$ -tuple integration in the anti-holomorphic sector. The contours are initially those of Felder’s, but following [48] one may deform them into

$$\int_{C_1} dt_1 \cdots \int_{C_n} dt_n \int_{S_1} ds_1 \cdots \int_{S_{l-n}} ds_{l-n}, \tag{112}$$

where  $C_i$  and  $S_i$  are as shown in figure 3. The chiral blocks (111) are then written by the standard Dotsenko–Fateev integrals.

These involve  $l$  screening operators in total and hence there are  $l + 1$  independent choices of contours depending on how many of them are in the holomorphic sector. As the configuration of the screening charges is now related to the Ishibashi states through (108), we expect that for each Ishibashi state there corresponds one of the  $l + 1$  independent solutions [47, 62] of the Knizhnik–Zamolodchikov equations. Let us see this in a specific example where  $j_1 = j_2 = 1/2$ ,  $m_1 = 1/2$  and  $m_2 = -1/2$ . In this case the number of the screening charges is  $l = 1$ , and hence  $n$  can be either 1 or 0. When  $n = 1$ , from (108) we see that  $\alpha = (0, 0, 0)$ , which is in the lattice sum of the A-type Ishibashi state  $\langle\langle A; j = 0 \rangle\rangle$ . Using the free-field formalism, we may express the chiral block as

$$I_{0,0}^A = \text{const} \times z_1^{\frac{1}{2(k+2)}} \bar{z}_1^{-\frac{3}{2(k+2)}} (1 - \xi)^{\frac{1}{2(k+2)}} \oint dt (z_1 - t)^{-\frac{1}{k+2}} t^{-\frac{1}{k+2}} (1 - \bar{z}_1 t)^{-\frac{1}{k+2}} \times \left\{ \frac{1}{t} + \frac{1}{z_1 - t} - \frac{\bar{z}_1}{1 - \bar{z}_1 t} \right\}, \tag{113}$$

where  $\xi \equiv z_1 \bar{z}_1$  is a cross ratio and we have used the global conformal invariance to set  $z_2 = \bar{z}_2 = 0$ . Changing the holomorphic contour into

$$\oint dt \rightarrow \int_0^{z_1} dt, \tag{114}$$

the chiral block is found in the form using a hypergeometric function

$$I_{0,0}^A = C_{0,0} \xi^{-\frac{3}{2(k+2)}} (1 - \xi)^{\frac{1}{2(k+2)}} F\left(\frac{1}{k+2}, \frac{k+1}{k+2}, \frac{k}{k+2}; \xi\right). \quad (115)$$

The normalization constant  $C_{0,0}$  is identified as the three-point coupling constant in the operator product expansion

$$\Phi_{1/2,1/2}(z, \bar{z}) \Phi_{1/2,-1/2}(0, 0) = C_{0,0} \Phi_{0,0}(0, 0) |z|^{-\frac{3}{k+2}} + C_{1,0} \Phi_{1,0}(0, 0) |z|^{\frac{1}{k+2}} \dots \quad (116)$$

In the off-boundary limit the chiral block agrees with the bulk two-point function. For  $n = 0$  we have one screening charge in the anti-holomorphic sector and (108) becomes  $\alpha = (-\sqrt{\frac{2}{k+2}}, 0, 0)$ . This boundary charge is included in the Ishibashi state  $\langle\langle A; j = 1 |$ , indicating the intermediate state of the chiral block being  $\Phi_{1,0}$ . The evaluation of the chiral block goes similarly as above, giving

$$I_{1,0}^A = C_{1,0} \xi^{\frac{1}{2(k+2)}} (1 - \xi)^{\frac{1}{2(k+2)}} F\left(\frac{3}{k+2}, \frac{k+3}{k+2}, \frac{k+4}{k+2}; \xi\right). \quad (117)$$

These results can be compared with the solutions of the Knizhnik–Zamolodchikov equations [62]. The chiral blocks (115) and (117) are two independent solutions which are now related to the Ishibashi states  $\langle\langle A; j = 0 |$  and  $\langle\langle A; j = 1 |$ , respectively. The two-point function on the disc is now written as a linear sum of these chiral blocks

$$\langle\langle \Phi_{1/2,1/2}(z_1, \bar{z}_1) \Phi_{1/2,-1/2}(z_2, \bar{z}_2) \rangle\rangle_{\text{disc}, \widetilde{A}, j} = I_{0,0}^A + \frac{S_{j1}}{S_{j0}} \sqrt{\frac{S_{00}}{S_{01}}} I_{1,0}^A. \quad (118)$$

There is another type of two-point function for isospin  $j = 1/2$ , but with  $m_1 = m_2 = 1/2$ . They are computed similarly as

$$\langle\langle \Phi_{1/2,1/2}(z_1, \bar{z}_1) \Phi_{1/2,1/2}(z_2, \bar{z}_2) \rangle\rangle_{\text{disc}, \widetilde{A}, j} = I_{0,1}^A + \frac{S_{j1}}{S_{j0}} \sqrt{\frac{S_{00}}{S_{01}}} I_{1,1}^A, \quad (119)$$

where the two chiral blocks are

$$I_{0,1}^A = C_{0,1} \xi^{\frac{2k+1}{2(k+2)}} (1 - \xi)^{\frac{1}{2(k+2)}} F\left(\frac{k+3}{k+2}, \frac{k+1}{k+2}, \frac{2k+2}{k+2}; \xi\right) \quad (120)$$

and

$$I_{1,1}^A = C_{1,1} \xi^{\frac{1}{2(k+2)}} (1 - \xi)^{\frac{1}{2(k+2)}} F\left(\frac{1}{k+2}, \frac{3}{k+2}, \frac{2}{k+2}; \xi\right). \quad (121)$$

The constants  $C_{0,1}$  and  $C_{1,1}$  are the three-point coupling constants of the bulk primaries

$$\Phi_{1/2,1/2}(z, \bar{z}) \Phi_{1/2,1/2}(0, 0) = C_{0,1} \Phi_{0,1}(0, 0) |z|^{-\frac{3}{k+2}} + C_{1,1} \Phi_{1,1}(0, 0) |z|^{\frac{1}{k+2}} \dots \quad (122)$$

#### 4.2. B-branes in $SU(2)_k$ model

The method discussed above may be applied to the B-branes without difficulty. Consider  $SU(2)_k$  B-branes constructed from A-branes by T-dualizing in the direction of  $U(1)_k$ , i.e. (98). They are characterized by  $\Lambda = \text{diag}(-1, 1, -1)$ . It is discussed in [10] that these B-branes couple only to primary fields with particular values of  $m$ . Let us see how this happens in the Coulomb-gas language. As the Cardy states  $\langle\langle B; \eta, j |$  are linear sums of the states  $\langle B(\alpha, \Lambda) |$ , we consider the one-point amplitude for a primary operator  $\Phi_{j,m}(z, \bar{z})$ ,

$$\langle\langle B(\alpha, \Lambda) | V_{j,m}(z) \bar{V}_{j,m}^\dagger(\bar{z}) (\text{screening operators}) | 0, 0; \alpha_0 \rangle\rangle, \quad (123)$$

where  $\Lambda = \text{diag}(-1, 1, -1)$ . The holomorphic and anti-holomorphic Coulomb-gas charges of the above one-point amplitude are respectively

$$-\alpha + \alpha_{j,m} + (\text{holomorphic screening charges}), \tag{124}$$

$$-4\alpha_0\rho - \Lambda^T \cdot \alpha + \alpha_{j,m}^\dagger + (\text{anti-holomorphic screening charges}). \tag{125}$$

The amplitude (123) is non-vanishing only when the net charge in each sector is neutral. Examining this condition using the expression of  $\alpha_{j,m}$  and  $\Lambda = \text{diag}(-1, 1, -1)$ , we find that the neutrality conditions are possible only when  $m = 0$  up to the reflection and periodic symmetry. Hence the B-branes couple to primary fields with  $m = 0$  (up to symmetry) only. We have used a particular vertex operator representation of the primary field from (39) here, but this property should be independent as all the vertex operator representations are equivalent modulo insertion of the screening operators.

Let us now turn to sample computations of two-point functions for the B-branes. For simplicity we focus on the  $j_1 = j_2 = 1/2$  case as in the previous subsection and restrict to  $k > 1$ . Using the same vertex operator representations (103) as A-branes, the chiral blocks are computed similarly, but this time with  $\Lambda = \text{diag}(-1, 1, -1)$ . The neutrality of Coulomb-gas charges constrains the values of the charges on the boundary and the configurations of the screening operators (i.e., how many of them are in the holomorphic/anti-holomorphic sectors). In the case of  $m_1 = 1/2$  and  $m_2 = -1/2$ , the neutrality conditions are satisfied in two cases: when  $\alpha = (0, 0, 0)$  and when  $\alpha = (-\sqrt{\frac{2}{k+2}}, 0, 0)$ . The former case corresponds to the intermediate state  $\Phi_{0,0}$  and the amplitude, including one holomorphic screening charge  $Q_1$ , is computed as

$$I_{0,0}^B = C_{0,0}\xi^{-\frac{3}{2(k+2)}}(1-\xi)^{\frac{3k+4}{2k(k+2)}}F\left(\frac{1}{k+2}, \frac{k+1}{k+2}, \frac{k}{k+2}; \xi\right). \tag{126}$$

In the latter case, which corresponds to intermediate  $\Phi_{1,0}$ , there is one screening charge  $\bar{Q}_1$  in the anti-holomorphic sector and we find the amplitude

$$I_{1,0}^B = C_{1,0}\xi^{\frac{1}{2(k+2)}}(1-\xi)^{\frac{3k+4}{2k(k+2)}}F\left(\frac{3}{k+2}, \frac{k+3}{k+2}, \frac{k+4}{k+2}; \xi\right). \tag{127}$$

The constants  $C_{0,0}$  and  $C_{1,0}$  are the same as those for the A-brane cases. The two-point function for B-type Cardy states are then written as linear combinations of these chiral blocks; using (98) we find

$$\langle \Phi_{1/2,1/2}(z_1, \bar{z}_1)\Phi_{1/2,-1/2}(z_2, \bar{z}_2) \rangle_{\text{disc}, \widetilde{B}; j, \eta} = I_{0,0}^B + \frac{S_{j1}}{S_{j0}}\sqrt{\frac{S_{00}}{S_{01}}}I_{1,0}^B. \tag{128}$$

In the off-boundary limit this has the same asymptotic behaviour as the A-brane counterpart. When  $m_1 = m_2 = 1/2$ , the neutrality condition of the charges can never be satisfied for the B-branes, marking a sharp contrast to the A-brane case. This indicates that the two-point function for the B-branes identically vanishes,

$$\langle \Phi_{1/2,1/2}(z_1, \bar{z}_1)\Phi_{1/2,1/2}(z_2, \bar{z}_2) \rangle_{\text{disc}, \widetilde{B}; \eta, j} = 0. \tag{129}$$

This result is reasonable as the fusion of  $\Phi_{1/2,1/2}$  and  $\Phi_{1/2,1/2}$  generates  $m = 1$  states which do not couple to B-branes.

### 5. Discussion

In this paper we have discussed free-field realization of  $SU(2)_k$  and parafermion boundary states based on the Coulomb-gas picture and presented a method to compute correlation

functions involving such boundaries. The formalism naturally includes the so-called B-type boundaries of both  $SU(2)_k$  and parafermion models. The examples of chiral blocks computed for  $SU(2)_k$  A-branes are solutions of the Knizhnik–Zamolodchikov equations, and it can be checked that the obtained correlation functions have appropriate clustering properties. Although we have not given explicit examples in this paper, the technique for finding exact correlation functions in the disc topology also applies to the  $\mathbb{Z}_k$  parafermion theory (with the primary fields of (41)). As the  $N = 2$  supersymmetric minimal models [63–69] differ from the  $SU(2)_k$  WZNW model only by the  $U(1)_k$  compactification radius, we expect that the same technique should also apply to these models without much difficulty.

We comment on similarity between the boundary states of the  $SU(2)_k$  model and those of the critical 3 state Potts model, which is a spin system having a  $\mathbb{Z}_3$  symmetry and is described by a CFT at  $c = 4/5$  [70]. The Potts model is known to have a  $W_3$  symmetry and is diagonal with respect to this  $W$ -algebra symmetry. There are six conformally invariant boundary states which also conserve the  $W$ -symmetry, and they are identified as three fixed and three mixed boundary conditions of the spin system [16]. In addition this model has two boundary states which conserve the conformal symmetry but break the  $W$ -symmetry. These two boundary states are interpreted to represent the free boundary condition and the so-called new boundary condition, which is found in [71]. It has been shown in [72] that these eight states are complete in the sense of [20]. Free-field representations of the Potts model boundary states are constructed and correlation functions involving such boundaries are computed in [27]. The obtained correlation functions are all consistent with the physics of the spin system. The free-field representation of the Potts model consists of two bosons and in [27] twelve boundary states are constructed, six of them corresponding to  $\det \Lambda = 1$  and  $W$ -conserving, the other six to  $\det \Lambda = -1$  and  $W$ -breaking, where  $\Lambda$  is a matrix similar to ours but is  $2 \times 2$  for the Potts model. Out of the six  $W$ -breaking boundaries four of them decouple from the rest of the theory, and only the two—corresponding to the free and new boundary conditions—can be seen in the analysis of [71, 72]. Clearly this situation is familiar to us; the  $W$ -breaking boundaries correspond to B-branes, and apart from some special cases (‘free’ and ‘new’ in the Potts model,  $m = 0$  and its identification images in the  $SU(2)_k$  model), the symmetry breaking boundaries decouple from the rest of the theory. This of course is not surprising since, as is well known, the three-state Potts model can be seen as the  $\mathbb{Z}_3$  parafermions.

By construction the boundary states we have formulated in this paper give desired overlaps. Besides, so far as we have checked the Coulomb-gas computation yields reasonable correlators on the disc topology. There are, however, a few subtle details in the formulation of boundary states which have not been settled in this paper. Firstly, we have considered states on the non-chiral Hilbert spaces which are simply direct products of left and right,  $\mathcal{H} \otimes \tilde{\mathcal{H}}$ . In the standard definition [16, 17], however, the Ishibashi states are built on asymmetric Hilbert spaces

$$\mathcal{H} \otimes U\tilde{\mathcal{H}}, \quad (130)$$

where  $U$  is an antiunitary operator. This operator  $U$  acts non-trivially on the  $SU(2)$  currents, and as a consequence our expressions of the gluing conditions (80) differ from some of the standard literature. If we wish we could have, for example, defined the operator  $\Lambda$  (and also  $\omega$ ) as antilinear, rather than linear, and adopted definitions of non-chiral Hilbert spaces which are closer to the traditional one (130). The second subtle point in our formulation is the BRST structure on which the lattice sums are based; in the three boson formulation of the  $SU(2)_k$  model which we have used, so far as we know there seems to be no proof in the literature of the Felder’s theorem, that is, triviality of the cohomology space on  $F_{r,s}$  except  $(r, s) = (l, n)$  that was needed to derive (49). The technique described in this paper would

be generalizable, for example, to  $SU(N)$  cases but for higher  $N$  the BRST structure should be more and more complicated. Thirdly, although it is straightforward to write down integral expressions of correlation functions on the annular topology, i.e. with two boundaries, we have not checked if these expressions satisfy various constraints for consistency. It is certainly worthwhile investigating in this direction more carefully. Finally, once the basic Coulomb-gas formulation of D-branes has been established, there are numerous more general directions to explore. We hope to come back to these issues in future publications.

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